1/14/26

- New people need the handout?
- All my lecture notes will be available on the webpage!
- No office hours tomorrow Jan. 15 (winter storm)

Last time, we did this example: (replace "equation" by "row" and "variable" by "column")

$$\left|\begin{array}{l} 2x+3y=8 \\ 3x+y=5 \end{array}\right| \qquad \left[\begin{array}{cc|c} 2 & 3 & 8 \\ 3 & 1 & 5 \end{array}\right]$$

algebraic operations with equations

same operations but in matrix form

$$\left|\begin{array}{l} x+0y=1 \\ 0x+y=2 \end{array}\right| \quad \text{i.e.} \quad \begin{array}{l} x=1 \\ y=2 \end{array} \qquad \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right]$$

this is a very simple example of a matrix in reduced row-echelon form

From this read solution

$x=1,\ y=2$

A little background on matrices. Need some terminology + notation. See §1.2

(Note - matrices will be very important this semester, not only in the above context!)

Def A rectangular array of numbers is a matrix.

Ex 1
$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 7 & 9 & 5 & 4 \end{bmatrix}$$

This has 3 rows and 4 columns

we refer to it as a 3×4 matrix

\# rows　　\# columns

The entries of the matrix are the numbers that appear.

e.g. the (3,2)-entry is 9

row　column

The entries aren't restricted to numbers. They can be variables of different kinds

Ex 2
$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

If this represents the matrix in Ex 1 then $a_{32} = 9$

A is a 3×4 matrix

Some terminology.

Def Let
$$B = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

B is an <u>$m \times n$ matrix</u>

If $m = n$, B is a <u>square</u> matrix

If $a_{ij} = 0$ for all i, j then B is a <u>zero matrix</u> (not necess. square)

Def Let B be a square matrix.

- if $a_{ij} = 0$ whenever $i \neq j$ then B is a <u>diagonal matrix</u>
- if $a_{ij} = 0$ whenever $i > j$ then B is an <u>upper triangular matrix</u>
- if $a_{ij} = 0$ whenever $i < j$ then B is a <u>lower triangular matrix</u>

Ex 3
$$A_1 = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \quad A_2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad A_3 = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix} \quad A_5 = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \quad A_6 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

$A_7 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ $A_8 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}$ $A_9 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $A_{10} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

square matrix : all <u>except</u> A_1, A_9

diagonal matrix : A_5, A_7, A_{10}

upper triangular matrix : A_3, A_5, A_6, A_7

lower triangular matrix: A_4, A_5, A_7, A_8, A_{10}

<u>Def</u> A matrix with only one column (ie an $m \times 1$ matrix for some m) is called a <u>column vector</u> or sometimes just a <u>vector</u>.

The set of all column vectors with n entries (ie n <u>components</u>) (ie the set of all $n \times 1$ matrices) is denoted $\mathbb{R}^n$.

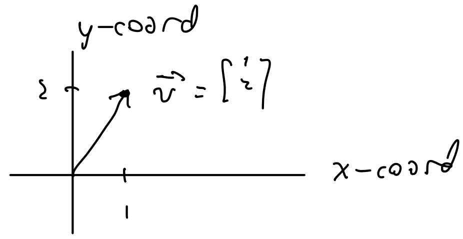
<u>Def</u> A matrix with only one row (a $1 \times n$ matrix) is a <u>row vector</u>

<u>Ex</u> $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ column vector, or just vector, in $\mathbb{R}^3$.

We write $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \in \mathbb{R}^3$.

$\mathbb{R}^1$, $\mathbb{R}^2$, $\mathbb{R}^3$, etc agree with what you've seen before. Just tweak the notation.

<u>Ex</u> $\vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \in \mathbb{R}^2$ $\longleftrightarrow$



Ex what's the set of all vectors in $\mathbb{R}^2$ of the form $\begin{bmatrix} a+1 \\ a+2 \end{bmatrix}$ look like?

think of a point (x,y) in $\mathbb{R}^2$ with such form

$$\begin{array}{l} x = a+1 \\ y = a+2 \end{array} \Rightarrow \begin{array}{l} a = x-1 \\ a = y-2 \end{array} \Rightarrow x-1 = y-2$$

$$\Rightarrow y = x+1$$

This is a line!

Back to linear systems.

Ex (see book §1.2)

$$\left| \begin{array}{l} 2x + 8y + 4z = 2 \\ 2x + 5y + z = 5 \\ 4x + 10y - z = 1 \end{array} \right|$$

$\begin{bmatrix} 2 & 8 & 4 \\ 2 & 5 & 1 \\ 4 & 10 & -1 \end{bmatrix}$ is the coefficient matrix

$\left[\begin{array}{ccc|c} 2 & 8 & 4 & 2 \\ 2 & 5 & 1 & 5 \\ 4 & 10 & -1 & 1 \end{array}\right]$ is the augmented matrix

Going thru similar computations to what we did Monday they reduce the augmented matrix to

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 11 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 3 \end{array}\right]$ ie $\begin{array}{l} x = 11 \\ y = -4 \\ z = 3 \end{array}$

This is an example of reduced row-echelon form

We write the sol'n as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ -4 \\ 3 \end{bmatrix}$$

see book for the steps

Ex Try same approach w/ 3 eqns, 5 variables (again from book §1.2)

$$\left|\begin{array}{l} x_1 - x_2 + 4x_5 = 2 \\ x_3 - x_5 = 2 \\ x_4 - x_5 = 3 \end{array}\right|$$

$$\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 4 & 2 \\ 0 & 0 & 1 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & -1 & 3 \end{array}\right]$$

another example of reduced row-echelon form

We can immediately read off the sol'n — solve for the circled variables

$$x_1 = 2 + x_2 - 4x_5$$

$$x_3 = 2 + x_5$$

$$x_4 = 3 + x_5$$

Now any values we (arbitrarily) assign to x_2 and x_5 gives a sol'n to the lin. syst!

e.g. if we choose $x_2 = 8$

and $x_5 = 10$

then $x_1 = 2 + 8 - 40 = -30$

$x_3 = 12$

$x_4 = 13$ ie $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -30 \\ 8 \\ 12 \\ 13 \\ 10 \end{bmatrix}$ is the corresponding solution

More generally

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 + r - 4t \\ r \\ 2 + t \\ 3 + t \\ t \end{bmatrix}$$

for any choice of r, t

– infinitely many sol'ns (sol'n set is 2-dimensional)